ZNOTES.ORG

UPDATED TO 2020-22 SYLLABUS

CAIE AS LEVEL
MATHS (9709)

SUMMARIZED NOTES ON THE MECHANICS SYLLABUS

1. Velocity and Acceleration

1.1. Kinematics Equations

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2 \qquad and \qquad s = vt - \frac{1}{2}at^2$$

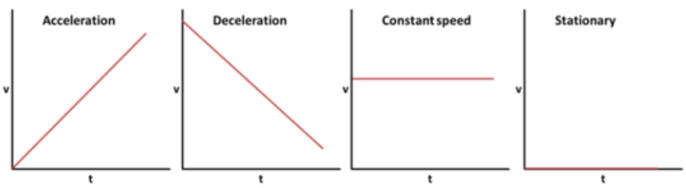
$$s = \frac{1}{2}(u + v)t$$

$$v^2 = u^2 + 2as$$

1.2. Displacement-Time Graph

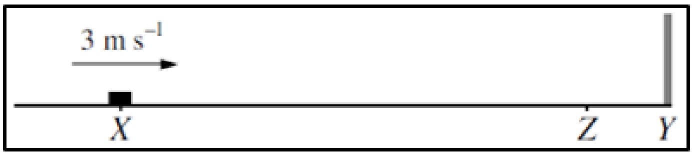
- Gradient = speed

1.3. Velocity-Time Graph



- Gradient = acceleration
- Area under graph = change in displacement

{S12-P42} Question 7:



The small block has mass $0.15kg$. The surface is horizontal. The frictional force acting on it is $0.12N$. Block set in motion from X with speed $3ms^{-1}$. It hits vertical surface at Y $2s$ later. Block rebounds from wall directly towards X and stops at Z. The instant that block hits wall it loses $0.072J$ of its kinetic energy. The velocity of the block from X to Y direction is v ms^{-1} at time t s after it leaves X.

i) Find values of v when the block arrives at Y and when it leaves Y. Also find t when block comes to rest at Z. Then sketch a velocity-time graph of the motion of the small block.

ii) Displacement of block from X, in the $\overrightarrow{\text{XY}}$ direction is s m at time t s. Sketch a displacement-time graph. On graph show values of s and t when block at Y and when it comes to rest at Z.

Solution

Part (i)

Calculating deceleration using Newton's second law:

$0.12 = 0.15a$ $a = \frac{0.12}{0.15} = 0.8ms^{-2}$

Calculate v at Y using relevant kinematics equation

$-0.8 = \frac{v-3}{2}$ $v = 1.4ms^{-1}$

Calculate kinetic energy at Y

$$E_K = \frac{1}{2}(0.15)(1.4)^2 = 0.147J$$

Calculate energy lost:

$$Initial - Change = Final$$

$$0.147 - 0.072 = 0.075J$$

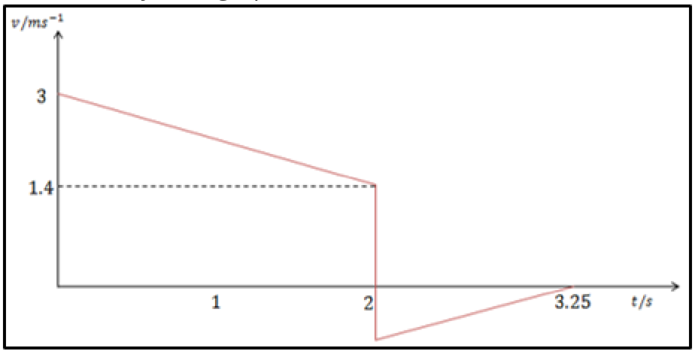
Calculate speed as leaving Y using k.E. formula:

$0.075 = \frac{1}{2}(0.15)v^2$ $v = 1ms^{-1}$

Calculate t when particle comes to rest:

$-0.8 = \frac{0-1}{t}$ $t = 1.25s$

Draw velocity-time graph with data calculated:



Part (ii)

Calculate displacement from X to Y

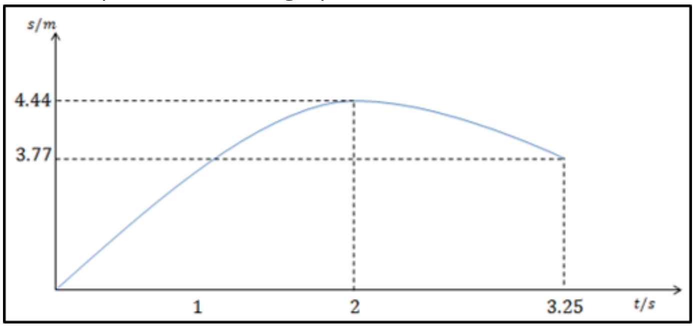
$s = (3 \times 2) + \frac{1}{2}(-0.8)(2)^2$ $s = 4.4m$

Calculate displacement from Y to Z

$$s = (1 \times 1.25) + \frac{1}{2}(-0.8)(1.25)^2$$

$$s = 0.625m \text{ } in \text{ } the \text{ } opposite \text{ } direction$$

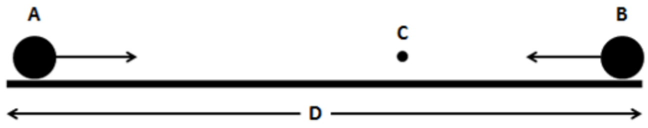
Draw displacement-time graph with data calculated:



1.4. Average Velocity

- For an object moving with constant acceleration over a period of time, these quantities are equal:
  - The average velocity
  - The mean of initial & final velocities
  - Velocity when half the time has passed

1.5. Relative Velocities



- Let s_A be the distance travelled by A and s_B for B

$$s_A = ut + \frac{1}{2}at^2 \qquad s_B = ut + \frac{1}{2}at^2$$

- If a collision occurs at point C

$$s_A + s_B = D$$

- This gives you the time of when the collision occurred
- Same analysis if motion is vertical

2. Force and Motion

Newton's 1st Law of Motion:
Object remains at rest or moves with constant velocity unless an external force is applied

Newton's 2nd Law of Motion:

$$F = ma$$

3. Vertical Motion

- ***Weight:*** *directly downwards*
- ***Normal contact force:*** *perpendicular to place of contact*

3.2. Common Results of Vertical Motion

Finding time taken to reach maximum height by a projectile travelling in vertical motion:

$$v = u + at$$

- *Let $v = 0$ and find t*
- *The time taken to go up and come back to original position would be double of this t*

Finding maximum height above a launch point use:

- $v^2 = u^2 - 2as$
- *Let $v = 0$ and find s*

Finding time interval for which a particle is above a given height:

- *Let the height be H and use*
- $s = ut + \frac{1}{2}at^2$
- *Let $s = H$*
- *There will be a quadratic equation in t*
- *Solve and find the difference between the 2 t's to find the time interval*

{S04-P04} Question 7:
Particle P_1 projected vertically upwards, from horizontal ground, with speed $30ms^{-1}$. At same instant P_2 projected vertically upwards from tower height $25m$, with speed $10ms^{-1}$

1. *Find the time for which P_1 is higher than the top of the tower*
2. *Find velocities of the particles at instant when they are same height*
3. *Find the time for which P_1 is higher than P_2 and moving upwards*

Solution

Part (i)
Substitute given values into displacement equation:

$$25 = (30)\,t + \frac{1}{2}(10)\,t^2$$

$$5t^2 + 30t - 25 = 0$$

Solve quadratic for t

$$t = 1s \text{ or } 5s$$

P_1 reaches tower at $t = 1$ then passes it again when coming down at $t = 5s$
Therefore, time above tower $= 5 - 1 = 4$ seconds
Part (ii)
Displacement of P_1 is S_1, and of P_2 is S_2 & relationship:

$$S_1 = 25 + S_2$$

Create equations for S_1 and S_2

$$S_1 = 30t + \frac{1}{2}(-10)\,t^2 \qquad S_2 = 10t + \frac{1}{2}(-10)\,t^2$$

Substitute back into initial equation

$$30t + \frac{1}{2}(-10)\,t^2 = 25 + 10t + \frac{1}{2}(-10)\,t^2$$

Simple cancelling

$$t = 1.25s$$

Find velocities

$$v = u + at$$

$$V_1 = 30 - 10(1.25) = 17.5ms^{-1}$$

$$V_2 = 10 - 10(1.25) = -2.5ms^{-1}$$

Part (iii)
We know when P_1 and P_2 at same height $t = 1.25s$. Find time taken to reach max height for P_1

$$v = u + at$$

V *is* 0 *at max height*
$0 = 30 - 10t \quad t = 3s$
Time for P_1 above $P_2 = 3 - 1.25 = 1.75$ seconds

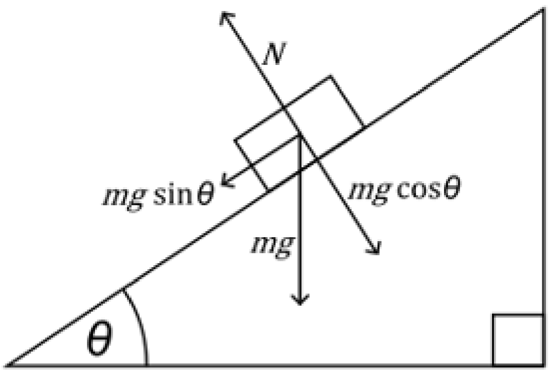
4. Resolving Forces

- *If force F makes an angle θ with a given direction, the effect of the force in that direction is $F\cos\theta$*

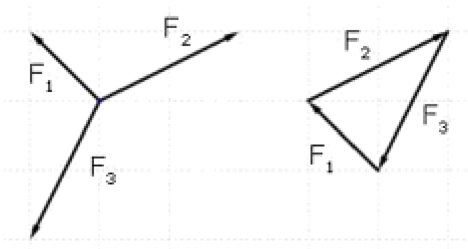
$$F\cos(90 - \theta) = F\sin\theta$$

$$F\sin(90 - \theta) = F\cos\theta$$

- *Forces in equilibrium: resultant = 0*
- *If drawn, forces will form a closed polygon*

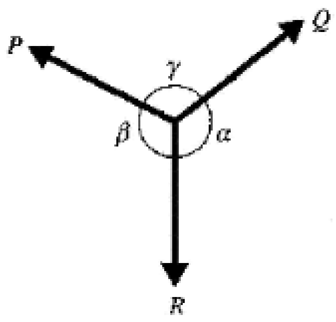


- ***Methods of working out forces in equilibrium:***
  - *Construct a triangle and work out forces*
  - *Resolve forces in x and y directions; sum of each = 0*



Lami's Theorem:

- *For any set of three forces P,Q and R in equilibrium*



$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

5. Friction

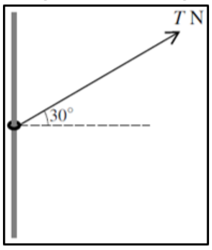
$$\text{Friction} = \text{Coefficient of Friction} \times \text{Normal Contact F}$$

$$F = \mu r$$

- *Friction always acts in the opposite direction of motion*
- *Limiting equilibrium: on the point of moving, friction at max (limiting friction)*
- *Smooth contact: friction negligible*
- *Contact force:*
  - *Refers to both F and N*
  - *Horizontal component of Contact force = F*
  - *Vertical component of Contact force = N*
  - *Magnitude of Contact force given by the formula:*

$$C = \sqrt{F^2 + N^2}$$

{W11-P43} Question 6:
The ring has a mass of $2kg$. The horizontal rod is rough and the coefficient of friction between ring and rod is 0.24. Find the two values of T for which the ring is in limiting equilibrium



Solution
The ring is in limiting equilibrium in two different scenarios; we have to find T in both:
Scenario 1: ring is about to move upwards

$$Resultant = T\sin 30 - friction - Weight\ of\ Ring$$

Since the system is in equilibrium, resultant = 0:

$$Contact\ Force = T\cos 30$$

$$\therefore Friction = 0.24 \times T\cos 30$$

Substitute relevant information in to initial equation

$$0 = T\sin 30 - 0.24T\cos 30 - 20$$

$$T = 68.5N$$

Scenario 2: ring is about to move downwards
This time friction acts in the opposite direction since friction opposes the direction of motion, thus:

$$Resultant = T\sin 30 + Friction - Weight\ of\ Ring$$

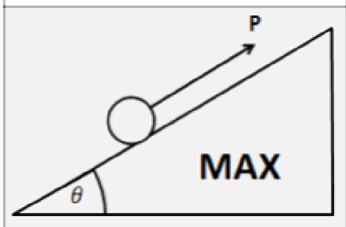
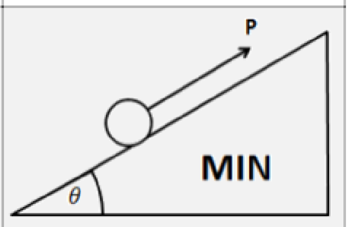
Using information from before:

$$0 = T\sin 30 + 0.24T\cos 30 - 20$$

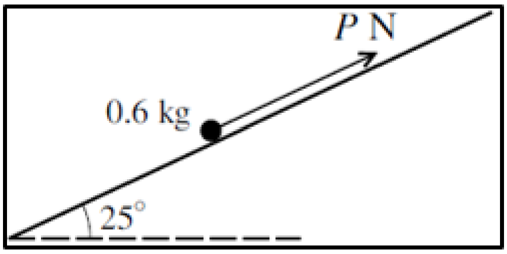
$$T = 28.3N$$

5.2. Equilibrium

Force required to keep a particle in equilibrium on a rough plane

Max Value	Min Value
	
The particle is about to move upThus, friction force acts down the slope $P = F + mgsin\theta$	The particle is about slip downThus, frictional force acts up the slope$F + P = mgsin\theta$

{W12-P43} Question 6:



Coefficient of friction is 0.36 *and the particle is in equilibrium. Find the possible values of* P

Solution

The magnitude of friction on particle in both scenarios is the same but acting in opposite directions
Calculate the magnitude of friction first:

$$Contact\ Force = 6\cos 25$$

$$\therefore Friction = 0.36 \times 6\cos 25$$

Scenario 1: particle is about to move upwards

$$P = 6\sin 25 + Friction$$

$$P = 4.49N$$

Scenario 2: particle is about to move downwards

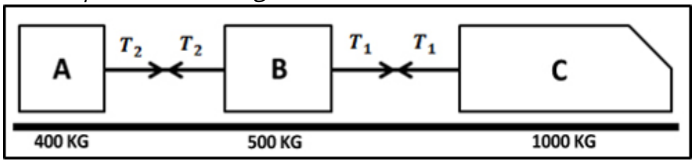
$$P = 6\sin 25 - friction$$

$$P = 0.578N$$

6. Connected Particles

Newton's 3rd Law of Motion:
For every action, there is an equal and opposite reaction

Question:
A train pulls two carriages:



The forward force of the engine is $F = 2500N$*. Find the acceleration and tension in each coupling. The resistance to motion of A, B and C are 200, 150 and 90N respectively.*

Solution:

To find acceleration, regard the system as a single object. The internal T*s cancel out and give:*

$$2500 - (200 + 150 + 90) = 1900a$$

$$\therefore a = 1.08ms^{-2}$$

To find T_1*, look at C*

$$F - T_1 - 200 = 1000a$$

$$2500 - T_1 - 200 = 1000 \times 1.08$$

$$T_1 = 1220N$$

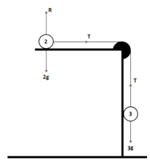
To find T_2*, look at A*

$$T_2 - 90 = 400a$$

$$T_2 - 90 = 400 \times 1.08$$

$$T_1 = 522N$$

6.2. Pulleys



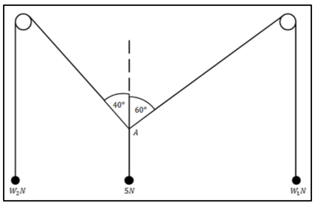
- *Equation 1:*

$$\therefore T = 2a$$

- *Equation 2:*

$$3g - T = 3a$$

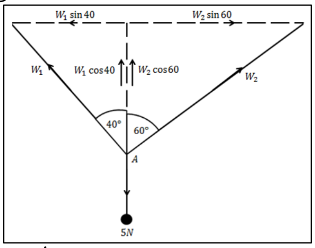
{W05-P04} Question 3:



The strings are in equilibrium. The pegs are smooth. All the weights are vertical. Find W_1 and W_2

Solution

Diagram showing how to resolve forces:



Resolving forces at A vertically:

$$W_1 \cos 40 + W_2 \cos 60 = 5$$

Resolving forces at A horizontally:

$$W_1 \sin 40 = W_2 \sin 60$$

Substitute second equation into first:

$$\left(\frac{W_2 \sin 60}{\sin 40}\right) \cos 40 + W_2 \cos 60 = 5$$

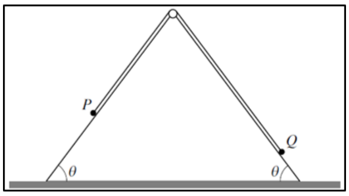
Solve to find W_2:

$$W_2 = 3.26N$$

Put this value back into the first equation to find W_1

$$W_1 = 4.40N$$

{S12-P41} Question 6:



P has a mass of $0.6kg$ and Q has a mass of $0.4kg$. The pulley and surface of both sides are smooth. The base of triangle is horizontal. It is given that $\sin \theta = 0.8$. Initially particles are held at rest on slopes with string taut. Particles are released and move along the slope

1. *Find tension in string. Find acceleration of particles while both are moving.*
2. *Speed of P when it reaches the ground is $2ms^{-1}$. When P reaches the ground, it stops moving. Q continues moving up slope but does not reach the pulley. Given this, find the time when Q reaches its maximum height above ground since the instant it was released*

Solution

Part (i)

Effect of weight caused by P in direction of slope:
Effect of weight $= mg \sin \theta$ where $\sin \theta = 0.8$
Effect of weight $= 4.8N$
Effect of weight caused by Q in direction of slope:
Effect of weight $= 0.4 \times 10 \times 0.8 = 3.2N$
Body P has greater mass than body Q so when released P moves down Q moves up on their slopes $\therefore$

$$4.8 - T = 0.6a$$

$$T - 3.2 = 0.4a$$

Solve simultaneous equations:
$\frac{4.8-T}{0.6} = \frac{T-3.2}{0.4}$ $T = 3.84N$
Substitute back into initial equations to find a:
$4.8 - 3.84 = 0.6a$ $a = 1.6ms^{-2}$

Part (ii)

Use kinematics equations to find the time which it takes P to reach the ground:
$a = \frac{v-u}{t}$ *and* $t = \frac{2-0}{1.6}$

$$t_1 = 1.25s$$

When P reaches the ground, only force acting on Q is its own weight in the direction of slope $= 3.2N$
$F = ma$ $-3.2 = 0.4a$

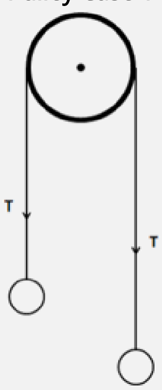
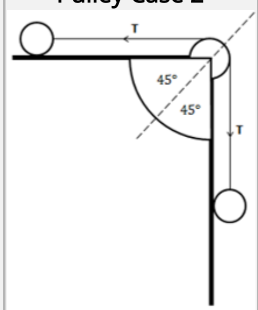
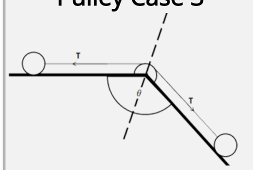
$$a = -8ms^{-2}$$

Now calculate the time taken for Q to reach max height
This occurs when its final velocity is 0.

$$-8 = \frac{0-2}{t} \qquad t_2 = 0.25s$$

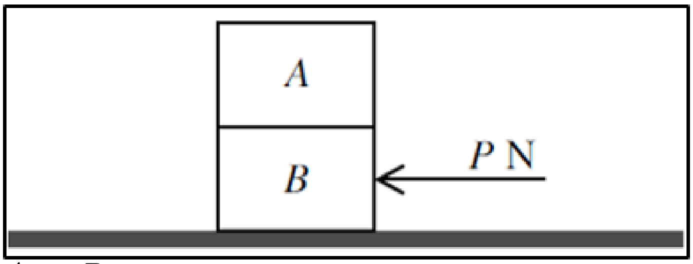
Now do simple addition to find total time:
Total Time $= 1.25 + 0.25 = 1.5s$

6.3. Force Exerted by String on Pulley

Pulley Case 1	Pulley Case 2	Pulley Case 3
		
Force on pulley = *2T* **Acts:** downwards	Force on pulley = *T√2* **Acts:** along dotted line	Force on pulley = *2T cos(1/2 θ)* **Acts:** inwards along dotted line which bisects θ

6.4. Two Particles

{S10-P43} Question 7:



A and B are rectangular boxes of identical sizes and are at rest on rough horizontal plane. A mass $= 200kg$ and B mass $= 250kg$. If $P \leq 3150$ boxes remain at rest. If $P > 3150$ boxes move

1. *Find coefficient of friction between B and floor*
2. *Coefficient of friction between boxes is 0.2. Given that $P > 3150$ and no sliding occurs between boxes. Show that the acceleration of boxes is not greater than $2ms^{-2}$*
3. *Find the maximum possible value of P in the above scenario*

Solution

Part (i)

$$F = \mu N$$

$F =$ to max P that does not move the boxes
*$N =$ to contact force of **both** boxes acting on floor*

$$\therefore 3150 = \mu \times (2000 + 2500)$$

$$\mu = 0.7$$

Part (ii)
Find frictional force between A and B:
$F = 0.2 \times 2000$ $F = 400N$
Use Newton's Second Law of Motion to find max acceleration for which boxes do not slide (below F)
$400 = 200a$ $a = 2ms^{-2}$
Part (iii)
P has to cause an acceleration of $2ms^{-2}$ on B which will pass on to A as they are connected bodies
Simply implement Newton's Second Law of Motion

$$\therefore P = (200 + 250)\,(2) + 3150$$

The 3150 comes from the force required to overcome the friction
$P = 900 + 3150$ $P = 4050N$

7. Work, Energy and Power

Principle of Conservation of Energy
Energy cannot be created or destroyed, it can only be changed into other forms

Work Done: $W = Fs$
Kinetic Energy: $E_k = \frac{1}{2}mv^2$
Gravitational Potential Energy: $E_P = mgh$
Power: $P = \frac{\mathrm{W}}{T}$ *and* $P = Fv$

7.2. Changes in Energy

$$\varepsilon_f - \varepsilon_i = (\mathrm{Work})_{engine} - (Work)_{friction}$$

- *ε_f is the final energy of the object*
- *ε_i is the initial energy of the object*
- *$(\mathrm{Work})_{\mathrm{engine}}$ is the energy caused by driving force acting on the object*
- *$(\mathrm{Work})_{\mathrm{friction}}$ is the energy used up by frictional force or any resistive force*

{S05-P04} Question 7:
Car travelling on horizontal straight road, mass 1200kg. Power of car engine is $20kW$ and constant. Resistance to motion of car is $500N$ and constant. Car passes point A with speed $10ms^{-1}$. Car passes point B with speed $25ms^{-1}$. Car takes $30.5s$ to move from A to B.

1. *Find acceleration of the car at A*
2. *Find distance AB by considering work & energy*

Solution:

Part (i)
Use formula for power to find the force at A

$$P = Fv$$

$$20000 = 10F \qquad Driving\ force = 2000N$$

We must take into account the resistance to motion

$$\therefore F = Driving\ Force - Resistance = 2000 - 500$$

$$F = 1500$$

Use Newton's Second Law to find acceleration:

$$1500 = 1200a \qquad a = \frac{1500}{1200} = 1.25ms^{-2}$$

Part (ii)
Use power formula to find work done by engine:

$$P = \frac{\mathrm{w.d.}}{t}$$

$20000 = \frac{\mathrm{w.d.}}{30.5}$ $\qquad w.d. = 610000J$
There is change in kinetic energy of the car so that means some work done by the engine was due to this:
$k.E.\ at\ A = \frac{1}{2}1200\,(10)^2$ $k.E.\ at\ B = \frac{1}{2}1200\,(25)^2$

$$Change\ in\ k.E. =\ k.E.\ at\ B - k.E.\ at\ A$$

$$Change\ in\ k.E. = 375000 - 60000 = 315000$$

There is also some work done against resistive force of $500N$*; due to law of conservation of energy, this leads us to the main equation:*

$$w.d.\ by\ engine = change\ in\ k.E + \ w.d.\ against\ resistance$$

$$610000 = 315000 + 500s$$

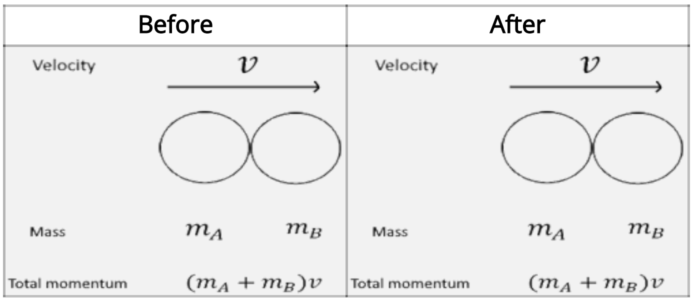
$$s = \frac{610000 - 315000}{500} = \frac{295000}{500} = 590m$$

8. Momentum

- *Momentum is a vector quantity, having the same direction as the velocity.*
- *The units of momentum are* Ns

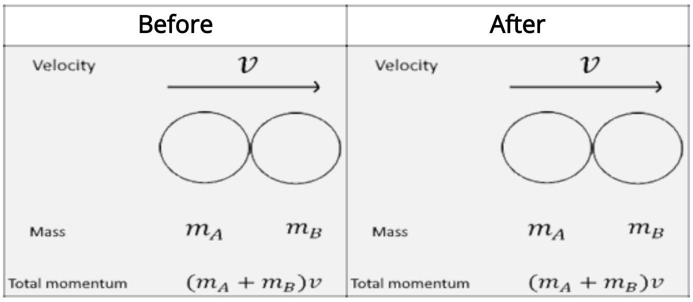
$$Momentum = mass \times velocity \qquad p = mv$$

- ***Principle of conservation of linear momentum:*** *when bodies in a system interact, total momentum remains constant provided no external force acts on the system.*



$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

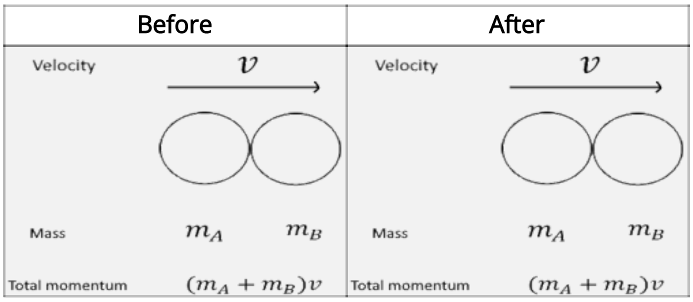
- *When both particles* A *and* B *move towards each other*



$$m_A u_A - m_B u_B = m_A v_A + m_B v_B$$

- *When both particles stick together, their velocity becomes the same after impact.*

Before	After



$$m_A u_A + m_B u_B = (m_A + m_B)v$$

{SP20-P04} Question 3:

Three small smooth spheres A, B and C of equal radii and of masses 4 kg, 2 kg and 3 kg respectively, lie in that order in a straight line on a smooth horizontal plane. Initially, B and C are at rest and A is moving towards B with speed 6ms^{-1}*. After the collision with B, sphere A continues to move in the same direction but with speed 2ms*$^{-1}$

i) Find the speed of B after this collision

ii) Sphere B collides with C. In this collision these two spheres coalesce to form an object D. Find the speed of D after this collision

Solution:

Part (i)

Calculate momentum of system before collision:

$$p = mv$$

$$p = (4 \times 6) + 2\,(0) + 3\,(0) = 24\ kgms^{-1}$$

Calculate momentum of system after collision:

$$p = (4 \times 2) + 2v$$

Apply conservation of momentum:

$$total\ momentum\ before = \ total\ momentum\ after$$

$24 = 8 + 2v\ v = \frac{16}{2} = 8.0\ ms^{-1}$

Part (ii)

Calculate momentum of system before collision:

$$p = (8 \times 2) + (3 \times 0) = 16\ kgms^{-1}$$

Calculate momentum of system after collision:

Note: The masses B and C combine to form D

$$p = (2 + 3)\,v$$

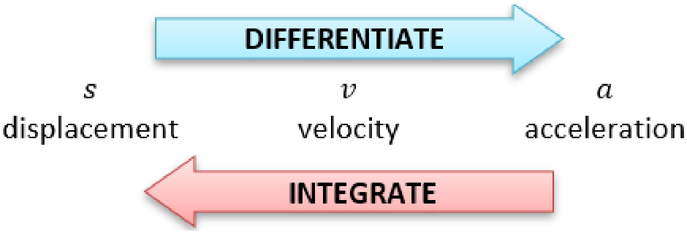
$$p = 5v$$

Apply conservation of momentum:

$$16 = 5v$$

$$v = \frac{16}{5} = 3.2\ ms^{-1}$$

9. General Motion in a Straight Line



- *Particle at instantaneous rest,* $v = 0$
- *Maximum displacement from origin,* $v = 0$
- *Maximum velocity,* $a = 0$

{W10-P42} Question 7:

Particle P *travels in straight line. It passes point* O *with velocity* $5ms^{-1}$ *at time* $t = 0s$*.*
P*'s velocity after leaving* O *given by:*

$$v = 0.002t^3 - 0.12t^2 + 1.8t + 5$$

v *of* P *is increasing when:* $0 < t < T_1$ *and* $t > T_2$
v *of* P *is decreasing when:* $T_1 < t < T_2$
i) Find the values of T_1 *and* T_2 *and distance* OP *when* $t = T_2$
ii) Find v *of* P *when* $t = T_2$ *and sketch velocity-time graph for the motion of* P

Solution

Part (i)

Find stationary points of v*; maximum is where* $t = T_1$ *and minimum is where* $t = T_2$

$$\frac{\mathrm{dv}}{\mathrm{dt}} = 0.006t^2 - 0.24t + 1.8$$

Stationary points occur where $\frac{\mathrm{dv}}{\mathrm{dt}} = 0$

$$\therefore 0.006t^2 - 0.24t + 1.8 = 0$$

Solve for t *in simple quadratic fashion:*

$$t = 30 \quad \textit{and} \quad 10$$

Naturally T_1 *comes before* T_2

$$\therefore T_1 = 10s \qquad \textit{and} \qquad T_2 = 30$$

Finding distance OP *by integrating*

$$\therefore s = \int_0^{30} v\mathrm{dt}$$

$$s = \int_0^{30} \left(0.002t^3 - 0.12t^2 + 1.8t + 5\right)\mathrm{dt}$$

$$s =_0^{30} \left[0.0005t^4 - 0.04t^3 + 0.9t^2 + 5t\right]$$

$$s = 285\ m$$

Part (ii)

Do basic substitution to find v

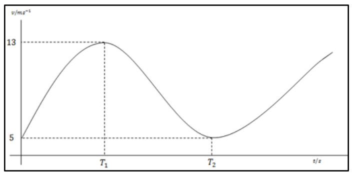
$$v = 0.002t^3 - 0.12t^2 + 1.8t + 5$$

$$t = 30 \qquad v = 5$$

To draw graph, find v *of* P *at* T_1 *using substitution and plot roughly*

$$\mathrm{v\ at}\ T_1 = 13$$

Graph:



{S13-P42} Question 6:

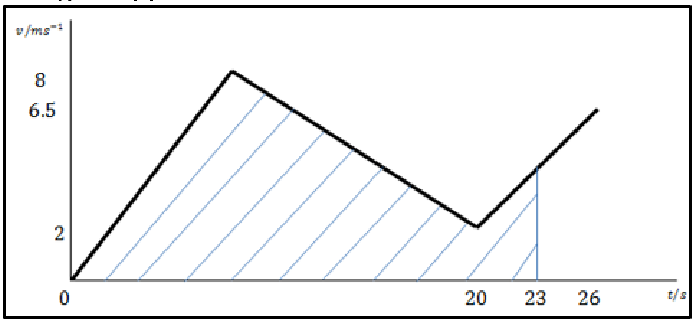
Particle P moves in a straight line. Starts at rest at point O *and moves towards a point* A *on the line. During first* 8 *seconds,* P*'s speed increases to* $8ms^{-1}$ *with constant acceleration. During next* 12 *seconds* P*'s speed decreases to* $2ms^{-1}$ *with constant deceleration.* P *then moves with constant acceleration for* 6 *seconds reaching point* A *with speed* $6.5ms^{-1}$

1. *Sketch velocity-time graph for* P*'s motion*
2. *The displacement of* P *from* O*, at time* t *seconds after* P *leaves* O*, is* s *metres. Shade region of the velocity-time graph representing* s *for a value of* t *where* $20 \leq t \leq 26$
3. *Show that for* $20 \leq t \leq 26$*,*

$$s = 0.375t^2 - 13t + 202$$

Solution:

Part (i) and (ii)



Part (ii)

First find s *when* $t = 20$*, this will produce a constant since* $20 \leq t \leq 26$

$$s_1 = \frac{1}{2}(8)(8) + \frac{1}{2}(8+2)(12) = 92m$$

Finding s *when* $20 \leq t \leq 26$*:*

$$s = ut + \frac{1}{2}at^2$$

Since the distance before 20 *seconds has already been taken into consideration:*

$$\mathbf{t = t - 20}$$

$$a = \frac{6.5 - 2}{6}$$

$$a = 0.75$$

$$\therefore s_2 = 2\left(t - 20\right) + \frac{1}{2}\left(0.75\right)\left(t - 20\right)^2$$

$$s_2 = 2t - 40 + 150 + 0.375t^2 - 15t$$

$$s_2 = 0.375t^2 - 13t + 110$$

Finally, add both to give you s

$$s = s_1 + s_2$$

$$s = 0.375t^2 - 13t + 110 + 92$$

$$s = 0.375t^2 - 13t + 202$$

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Maths (9709)